HEARING PSEUDOCONVEXITY WITH THE KOHN LAPLACIAN

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1. Introduction

Mark Kac's famous question "Can one hear the shape of a drum?" asks whether the spectrum of the Dirichlet Laplacian determines a planar domain up to congruence [Ka66]. This question was answered negatively by Gordon, Webb, and Wolpert(cf. [GWW92]). It has inspired a tremendous amount of research on the interplay of the spectrum of differential operators and the geometry of ambient spaces. Here we study the several complex variables analogue of Kac's question: To what extent is the geometry of a bounded domain Ω in $\mathbb{C}^n$ determined by the spectrum of the $\overline{\partial}$-Neumann and Kohn Laplacians? Since the work of Kohn [Ko63, Ko64], it has been discovered that various notions of regularity of the $\overline{\partial}$-Neumann and Kohn Laplacians, such as subellipticity, hypoellipticity, and compactness, are intimately related to the boundary geometry of the domain. (See, for example, the surveys [BSt99, Ch99, DK99, FS01].) It is then natural to expect that one should be able to "hear" more about the geometry of a bounded domain in $\mathbb{C}^n$ with the $\overline{\partial}$-Neumann and Kohn Laplacians than with the usual Dirichlet Laplacians. In this paper, we prove the following:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{C}^n$, $n > 1$, with connected Lipschitz boundary $b\Omega$. Let $\Box_{b,q}$ be the Kohn Laplacian on $L^2_{(0,q)}(b\Omega)$. Let $\operatorname{esspec}(\Box_{b,q})$ be the essential spectrum of $\Box_{b,q}$. If $\inf \operatorname{esspec}(\Box_{b,q}) > 0$ for all $1 \le q \le n-1$, then Ω is pseudoconvex.*

It was shown by Kohn [Ko86] that on smooth pseudoconvex boundaries $b\Omega$ in Stein manifolds, $\overline{\partial}_b$ has closed range in $L^2_{(0,q)}(b\Omega)$ for all $1 \le q \le n-1$. Independently, Shaw [Sh85] (for $1 \le q \le n-2$) and Boas-Shaw [BSh86] (for $q = n-1$) established L^2-existence theorems for the $\overline{\partial}_b$-operator on smooth pseudoconvex boundaries in $\mathbb{C}^n$. Recently, Shaw [Sh03] extended these results to pseudoconvex Lipschitz boundaries. In light of these results and Theorem 1.1, for connected and sufficiently smooth boundaries in $\mathbb{C}^n$, pseudoconvexity is characterized by positivity of the infimum of the spectrum (or the essential spectrum) of the Kohn Laplacians on all $(0,q)$-forms, $1 \le q \le n-1$.

This paper is organized as follows. In Section 2, we recall necessary setups and definitions. Section 3 contains the proof of Theorem 1.1. Further remarks are given in Section 4.

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2. Preliminaries

We first review the well-known operator theoretic setup (cf. [H65, FK72]). Let $T_k\colon H_k \to H_{k+1}$, $k = 1, 2$, be densely defined, closed operators between Hilbert spaces. Assume that $\mathcal{R}(T_1) \subset \mathcal{N}(T_2)$, where $\mathcal{R}$ and $\mathcal{N}$ denote the range and kernel of the operators. Let T_k^* be the Hilbert space adjoint of T_k. Then T_k^* is also densely defined and closed. Let

$$Q(u,v) = (T_1^* u, T_1^* v) + (T_2 u, T_2 v)$$

with $\mathrm{Dom}(Q) = \mathrm{Dom}(T_1^*) \cap \mathrm{Dom}(T_2)$. It is easy to see that $Q(u,v)$ is a non-negative, densely defined, closed sesquilinear form on H_2. It follows that $Q(u,v)$ uniquely determines a non-negative, densely defined, self-adjoint operator $\Box$ on H_2 such that $\mathrm{Dom}(\Box^{1/2}) = \mathrm{Dom}(Q)$ and $Q(u,v) = (\Box u, v)$ for all $u \in \mathrm{Dom}(\Box)$ and $v \in \mathrm{Dom}(Q)$. (We refer the reader to [D95, K76, RS] for detail on sesquilinear forms and self-adjoint operators.) The spectrum $\mathrm{spec}\,(\Box)$ of $\Box$ is a non-empty closed subset of $[0, \infty)$ and the infimum of the spectrum is given by

$$\inf \mathrm{spec}\,(\Box) = \inf\{Q(u,u); \quad u \in \mathrm{Dom}(Q), \|u\| = 1\}.$$

For any positive integer j, let

$$\lambda_j = \sup_{v_1,\ldots,v_{j-1} \in \mathrm{Dom}(Q)} \inf \left\{ Q(u,u);\ u \in \mathrm{Dom}(Q), u \perp v_i, 1 \le i \le j-1, \|u\| = 1 \right\}.$$

Then $\Box$ has compact resolvent if and only if $\lambda_j \to \infty$. In this case, λ_j is the j^{th} eigenvalue of $\Box$, when the eigenvalues are arranged in increasing order and repeated according to multiplicity. If $\Box$ has non-compact resolvent (equivalently, the essential spectrum $\mathrm{esspec}\,(\Box)$ is non-empty), λ_j is either an eigenvalue of finite multiplicity or the bottom of $\mathrm{esspec}\,(\Box)$. In either cases, $\lim_{j\to\infty} \lambda_j = \inf \mathrm{esspec}\,(\Box)$. In what follows, we will set $\inf \mathrm{esspec}\,(\Box) = \infty$ when $\mathrm{esspec}\,(\Box)$ is empty.

Lemma 2.1. *With the above notations and assumptions,* $\inf \mathrm{spec}\,(\Box) > 0$ *if and only if* $\mathcal{R}(T_2)$ *is closed and* $\mathcal{R}(T_1) = \mathcal{N}(T_2)$. *Furthermore,* $\inf \mathrm{esspec}\,(\Box) > 0$ *if and only if there exists a finite dimensional subspace* $L \subset \mathrm{Dom}(Q)$ *such that* $\mathcal{R}(T_2\big|_{L^\perp})$ *is closed and* $\mathcal{R}(T_1) \cap L^\perp = \mathcal{N}(T_2) \cap L^\perp$.

The first part of the lemma is well-known (compare [H65], Theorem 1.1.2; [C83], Proposition 3; and [Sh92], Proposition 2.3). We provide a proof here for completeness. To prove the forward direction, we note that $\inf \mathrm{spec}\,(\Box) > 0$ implies that $\Box$ has a bounded inverse N defined on all H_2. Hence each $u \in H_2$ has an orthogonal decomposition $u = T_1 T_1^* N u + T_2^* T_2 N u$. It follows that $\mathcal{R}(T_1) = \mathcal{N}(T_2)$ and $\mathcal{R}(T_2^*) = \mathcal{N}(T_1^*)$. Since now T_2^* has closed range, so is T_2. We thus conclude the prove of forward direction. To prove the opposite, for any $u \in \mathrm{Dom}(Q)$, we write $u = u_1 + u_2$ where $u_1 \in \mathrm{Dom}(Q) \cap \mathcal{N}(T_2)$ and $u_2 = \mathrm{Dom}(Q) \cap \mathcal{N}(T_2)^\perp$. Since $\mathcal{N}(T_2) = \mathcal{R}(T_1) = \mathcal{N}(T_1^*)^\perp$ and $\mathcal{N}(T_2)^\perp = \mathcal{R}(T_1)^\perp = \mathcal{N}(T_1^*)$, there exists a positive constant C such that $\|u\|^2 = \|u_1\|^2 + \|u_2\|^2 \le C(\|T_1^* u_1\|^2 + \|T_2 u_2\|^2) = CQ(u,u)$. This concludes the proof of the backward direction.

For a proof of the second part of the lemma, we observe that by the above-mentioned spectral theoretic results, $\inf \mathrm{esspec}\,(\Box) > 0$ if and only if there exists a positive constant C and a finite dimensional subspace L of $\mathrm{Dom}(Q)$ such that

$$Q(u,u) \ge C\|u\|, \quad u \in \mathrm{Dom}(Q) \cap L^\perp.$$

To prove the forward direction, let $H_2' = H_2 \ominus L$ and let $T_2' = T_2\big|_{H_2'}$ and $T_1^{*\prime} = T_1^*\big|_{H_2'}$. Then $T_2'\colon H_2' \to H_3$ and $T_1^{*\prime}\colon H_2' \to H_1$ are densely defined, closed operators. Let $T_1'\colon H_1 \to H_2'$ be the adjoint of $T_1^{*\prime}$. It is easy to see that $\mathcal{R}(T_1') \subset \mathcal{N}(T_2')$ and $\mathrm{Dom}(T_1') = \mathrm{Dom}(T_1)$.

Applying the first part of the lemma to the operators $T_1'\colon H_1 \to H_2'$ and $T_2'\colon H_2' \to H_3$ and the sesquilinear form

$$Q'(u,v) = (T_1'^{*}u, T_1'^{*}v) + (T_2'u, T_2'v)$$

on H_2' with $\mathrm{Dom}(Q') = \mathrm{Dom}(Q) \cap L^\perp$, we obtain that T_1' and T_2' have closed range and $\mathcal{R}(T_1') = \mathcal{N}(T_2')$. We then conclude the proof of the forward direction by noting that $\mathcal{R}(T_1') = \mathcal{R}(T_1) \cap L^\perp$ and $\mathcal{N}(T_2') = \mathcal{N}(T_2) \cap L^\perp$. The converse is treated similarly as above and is left to the reader.

Remark. Let $\widetilde{H_2} = \mathcal{N}(T_1^*)^\perp$. Let $\widetilde{T}_1^* = T_1^*|_{\widetilde{H}_2}$ and let $\widetilde{Q}(u,v) = (\widetilde{T}_1^* u,\ \widetilde{T}_1^* v)$ be the sesquilinear form on $\widetilde{H}_2$ with $\mathrm{Dom}(\widetilde{Q}) = \mathrm{Dom}(T_1^*) \cap \widetilde{H}_2$. Let $\widetilde{\Box}$ be the self-adjoint operator determined by $\widetilde{Q}(u,v)$. In this case, $\inf\mathrm{spec}\,(\widetilde{\Box}) > 0$ if and only if $\mathcal{R}(T_1) = \mathcal{N}(T_1^*)^\perp$, and $\inf\mathrm{esspec}\,(\widetilde{\Box}) > 0$ if and only if there exists a finite dimensional subspace L of $\widetilde{H}_2$ such that $\mathcal{R}(T_1) \cap L^\perp = L^\perp$.

We now review the $\overline{\partial}_b$-complex as introduced by Kohn [Ko65, KR65], and adapted to Lipschitz boundaries by Shaw [Sh03]. Let Ω be a bounded Lipschitz domain in $\mathbb{C}^n$. (Recall that $b\Omega$ is Lipschitz if it is given locally by a Lipschitz graph.) Let $\rho \in \mathrm{Lip}\,(\mathbb{C}^n)$ be a defining function of $b\Omega$ such that $\rho < 0$ on Ω and $C_1 \le |d\rho| \le C_2$ a.e. on $b\Omega$ for some positive constants C_1 and C_2 (cf. [Sh03]). Let $I^{0,q}$, $0 \le q \le n$, be the ideal in $\Lambda^{0,q}T^*(\mathbb{C}^n)$ generated by ρ and $\overline{\partial}\rho$. Let $\Lambda^{0,q}T^*(b\Omega)$ be the orthogonal complement with respect to the standard Euclidean metric of $I^{0,q}|_{b\Omega}$ in $\Lambda^{0,q}T^*(\mathbb{C}^n)|_{b\Omega}$. Let $\tau\colon \Lambda^{0,q}T^*(\mathbb{C}^n)|_{b\Omega} \to \Lambda^{0,q}T^*(b\Omega)$ be the orthogonal projection.

Let $L^2_{(0,q)}(b\Omega)$ be the space of $(0,q)$-forms with L^2-coefficients, equipped with the induced Euclidean metric on $b\Omega$; that is, the projections under τ of $(0,q)$-forms on $\mathbb{C}^n$ whose coefficients are in $L^2(b\Omega)$ when restricted to $b\Omega$. The operator $\overline{\partial}_{b,q}\colon L^2_{(0,q)}(b\Omega) \to L^2_{(0,q+1)}(b\Omega)$, $0 \le q \le n-1$, defined in the sense of distribution as the restriction of $\overline{\partial}_q$ to the boundary $b\Omega$, is densely defined and closed (see [Sh03]). Let $\overline{\partial}^*_{b,q}$ be the Hilbert space adjoint of $\overline{\partial}_{b,q}$. Let

$$Q_{b,q}(u,v) = (\overline{\partial}_{b,q}u,\ \overline{\partial}_{b,q}v) + (\overline{\partial}^*_{b,q-1}u,\ \overline{\partial}^*_{b,q-1}v)$$

with $\mathrm{Dom}(Q_{b,q}) = \mathrm{Dom}(\overline{\partial}_{b,q}) \cap \mathrm{Dom}(\overline{\partial}^*_{b,q-1})$ when $1 \le q \le n-2$, and let

$$Q_{b,n-1}(u,v) = (\overline{\partial}^*_{b,n-2}u,\ \overline{\partial}^*_{b,n-2}v)$$

with $\mathrm{Dom}(Q_{b,n-1}) = \mathrm{Dom}(\overline{\partial}^*_{b,n-2}) \cap \mathcal{N}(\overline{\partial}^*_{b,n-2})^\perp$. Then $Q_{b,q}$, $1 \le q \le n-1$, are non-negative, closed, and densely defined sesquilinear forms on $L^2_{(0,q)}(b\Omega)$. Therefore it uniquely determines a non-negative, closed, densely defined, and self-adjoint operator $\Box_{b,q}$ on $L^2_{(0,q)}(b\Omega)$ such that $\mathrm{Dom}(\Box_{b,q}^{1/2}) = \mathrm{Dom}(Q_{b,q})$ and $Q_{b,q}(u,v) = (\Box_{b,q}u, v)$ for all $u \in \mathrm{Dom}(\Box_{b,q})$ and $v \in \mathrm{Dom}(Q_q)$. The Kohn Laplacian is formally given by $\Box_{b,q} = \overline{\partial}_{b,q-1}\overline{\partial}^*_{b,q-1} + \overline{\partial}^*_{b,q}\overline{\partial}_{b,q}$ for $1 \le q \le n-2$ and $\Box_{b,n-1} = \overline{\partial}_{b,n-2}\overline{\partial}^*_{b,n-2}|_{\mathcal{N}(\overline{\partial}^*_{n-2})^\perp}$. (Notice that on top degree $(0,n-1)$-forms, the Kohn Laplacian here is the restriction to the orthogonal complement of $\mathcal{N}(\overline{\partial}^*_{b,n-2})$ of the usual Kohn Laplacian. We make this restriction because the kernel of $\overline{\partial}^*_{b,n-2}$ is infinite dimensional.) We refer the reader to the monographs [FK72] and [CS01] for detail on the subject.

3. Proof of the Main Theorem

Let $\rho \in \mathrm{Lip}\,(\mathbb{C}^n)$ be a global defining function of Ω such that $\rho < 0$ on Ω and $C_1 \le |d\rho| \le C_2$ a.e. on $b\Omega$. Arguing via *reductio ad absurdum*, we assume that Ω is not pseudoconvex. Then there exists a domain $\widetilde{\Omega} \supsetneqq \Omega$ such that every holomorphic function on Ω extends holomorphically to $\widetilde{\Omega}$ (cf. [H91]). Since $b\Omega$ is Lipschitz, $\widetilde{\Omega} \setminus \mathrm{cl}\,(\Omega)$ is non-empty. After a translation and a unitary transformation, we may assume that the origin is in $\widetilde{\Omega} \setminus \mathrm{cl}\,(\Omega)$ and the z_n-axis has a non-empty intersection with Ω. Furthermore, we may assume that the positive y_n-direction is the outward normal direction of the intersection of the y_n-axis with $b\Omega$ and $b\Omega \cap \widetilde{\Omega}$ is parameterized near the intersection by $y_n = h(z_1, \ldots, z_{n-1}, x_n)$ for some Lipschitz function h.

For any integers $\alpha \ge 0$, $m \ge 1$, and $q \ge 1$, and for any $\{k_1, \ldots, k_{q-1}\} \subset \{1, 2, \ldots, n-1\}$, let

$$u_{\alpha,m}(k_1, \ldots, k_q) = \frac{(\alpha+q-1)!\bar{z}_n^{m\alpha}(\bar{z}_{k_1}\cdots\bar{z}_{k_q})^{m-1}}{r_m^{\alpha+q}} \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} d\bar{z}_{k_1} \wedge \ldots \wedge \widehat{d\bar{z}_{k_j}} \wedge \ldots \wedge d\bar{z}_{k_q}$$

where $k_q = n$, $r_m = |z_1|^{2m} + \ldots + |z_n|^{2m}$, and $\widehat{d\bar{z}_{k_j}}$ indicates as usual the omission of $d\bar{z}_{k_j}$ from the wedge product. It is evident that $u_{\alpha,m}(k_1, \ldots, k_q)$ is a smooth $(0, q-1)$-form on $\mathbb{C}^n \setminus \{0\}$ that is skew-symmetric with respect to the indices $(k_1, \ldots, k_{q-1})$. In particular, $u_{\alpha,m}(k_1, \ldots, k_q) = 0$ when two k_j's are identical. Write $K = (k_1, \ldots, k_q)$, $d\bar{z}_K = d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_q}$, $\bar{z}_K^{m-1} = (\bar{z}_{k_1} \cdots \bar{z}_{k_q})^{m-1}$, and $\widetilde{d\bar{z}_{k_j}} = d\bar{z}_{k_1} \wedge \ldots \wedge \widehat{d\bar{z}_{k_j}} \wedge \ldots \wedge d\bar{z}_{k_q}$. Then

$$\begin{aligned}
\overline{\partial} u_{\alpha,m}(k_1, \ldots, k_q) &= -\frac{(\alpha+q)!m\bar{z}_n^{m\alpha}\bar{z}_K^{m-1}}{r_m^{\alpha+q+1}} \Big(r_m d\bar{z}_K + \big(\sum_{\ell=1}^{n} \bar{z}_\ell^{m-1} z_\ell^m d\bar{z}_\ell\big) \wedge \big(\sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} \widetilde{d\bar{z}_{k_j}}\big)\Big) \\
&= -\frac{(\alpha+q)!m\bar{z}_n^{m\alpha}\bar{z}_K^{m-1}}{r_m^{\alpha+q+1}} \sum_{\ell \in \{1,\ldots,n\}\setminus\{k_1,\ldots,k_q\}} z_\ell^m \bar{z}_\ell^{m-1} \big(\bar{z}_\ell d\bar{z}_K + \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} \widetilde{d\bar{z}_{k_j}}\big) \\
&= m \sum_{\ell=1}^{n-1} z_\ell^m u_{\alpha,m}(\ell, k_1, \ldots, k_q).
\end{aligned}$$

In particular, $u_{\alpha,m}(1, \ldots, n)$ is $\overline{\partial}$-closed. Let $N = (1/|\partial\rho|)\sum_{j=1}^{n} \rho_{z_j} \partial/\partial\bar{z}_j$ and let

$$u^b_{\alpha,m}(k_1, \ldots, k_q) = \tau(u_{\alpha,m}(1, 2, \ldots, n)) = N \lrcorner \Big(\frac{\overline{\partial}\rho}{|\overline{\partial}\rho|} \wedge u_{\alpha,m}(k_1, \ldots, k_q)\Big) \in L^2_{(0,q-1)}(b\Omega),$$

where $\lrcorner$ denotes the contraction operator. Then for $1 \le q \le n-1$,

$$\overline{\partial}_{b,q-1} u^b_{\alpha,m}(k_1, \ldots, k_q) = m \sum_{\ell=1}^{n-1} z_\ell^m u^b_{\alpha,m}(\ell, k_1, \ldots, k_q).$$

We now show that $u^b_{\alpha,m}(1, 2, \ldots, n) \perp \mathcal{N}(\overline{\partial}^*_{b,n-1})$. Let $\star\colon L^2_{(p,q)}(\Omega) \to L^2_{(n-p,n-q)}(\Omega)$ be the Hodge star operator, defined by $\langle \phi, \psi \rangle dV = \phi \wedge \star\psi$ where dV is the Euclidean volume form. Let $v \in \mathcal{N}(\overline{\partial}^*_{b,n-1})$. Let $\theta = \star(dz_1 \wedge \ldots \wedge dz_n \wedge \overline{\partial}\rho/|\overline{\partial}\rho|)$. Then $v = \bar{f}\theta$ for some $f \in L^2(b\Omega)$ with $\overline{\partial}_b f = 0$. It follows from a version of Hartogs-Bochner extension theorem that there exists a holomorphic function F on Ω such that the non-tangential limit of F

agrees with f a.e. on $b\Omega$, and

$$\lim_{\epsilon\to 0^+}\int_{b\Omega}|F(z-\epsilon\nu(z))-f(z)|^2d\sigma=0$$

where $\nu(z)=\nabla\rho/|\nabla\rho|$. (See, for example, Theorem 7.1 in [Ky95]. Although the theorem is stated only for C^1-smooth boundaries, the proof works for Lipschitz boundaries with only minor modifications.) Let $\nu_\delta(z)$ be the convolution of $\nu(z)$ with appropriate Friederichs' mollifiers. Then there exists a subsequence $\delta_j\to 0$ such that $\nu_{\delta_j}(z)\to\nu(z)$ a.e. on $b\Omega$. Therefore,

$$\begin{aligned}(u^b_{\alpha,m}(1,\dots,n),v)&=\int_{b\Omega}f(z)u^b_{\alpha,m}(1,\dots,n)(z)\wedge dz_1\dots\wedge dz_n\\&=\lim_{\epsilon\to 0}\int_{b\Omega}F(z-\epsilon\nu(z))u^b_{\alpha,m}(1,\dots,n)(z)\wedge dz_1\dots\wedge dz_n\\&=\lim_{\epsilon\to 0}\lim_{\delta_j\to 0}\int_{b\Omega}F(z-\epsilon\nu_{\delta_j}(z))u^b_{\alpha,m}(1,\dots,n)(z)\wedge dz_1\dots\wedge dz_n\\&=\lim_{\epsilon\to 0}\lim_{\delta_j\to 0}\int_{\Omega}\overline{\partial}\big(F(z-\epsilon\nu_{\delta_j}(z))u_{\alpha,m}(1,\dots,n)(z)\wedge dz_1\dots\wedge dz_n\big)=0.\end{aligned}$$

Hence $u^b_{\alpha,m}(1,\dots,n)\perp\mathcal{N}(\overline{\partial}^*_{b,n-1})$ as claimed.

By Lemma 2.1 and the subsequence remark, we can choose a sufficiently large positive integer M such that there exist subspaces S_q of $\mathrm{Dom}(Q_{b,q})$ for $1\le q\le n-2$ and S_{n-1} of $\mathcal{N}(\overline{\partial}^*_{b,n-2})^\perp$, all of which have dimensions $<M$ and satisfy $\mathcal{R}(\overline{\partial}_{b,q-1})\cap S_q^\perp=\mathcal{N}(\overline{\partial}_{b,q})\cap S_q^\perp$, $1\le q\le n-2$, and $\mathcal{R}(\overline{\partial}_{b,n-2})\cap S_{n-1}^\perp=S_{n-1}^\perp$. Fix $m\ge 1$ (to be specified later) and let $\mathcal{F}_0$ be the linear span of $\{u^b_{\alpha,m}(1,\dots,n);\ \alpha=1,\dots,M^{n-1}\}$. For any $u\in\mathcal{F}_0$ and for any $\{k_1,\dots,k_{q-1}\}\subset\{1,\dots,n-1\}$, we set

$$u(k_1,\dots,k_{q-1},n)=\sum_{j=1}^k c_j u^b_{\alpha_j,m}(k_1,\dots,k_{q-1},n)$$

if $u=\sum_{j=1}^k c_j u^b_{\alpha_j,m}(1,\dots,n)$. We decompose $\mathcal{F}_0$ into a direct sum of M^{n-2} subspaces, each of which is M-dimensional. Since $\dim(S_{n-1})<M$ and $u_{\alpha,m}(1,\dots,n)\in\mathcal{N}(\overline{\partial}^*_{b,n-2})^\perp$, there exists a non-zero form u in each of the subspaces such that $\overline{\partial}_b v_u(\emptyset)=u$ for some $v_u(\emptyset)\in L^2_{(0,n-2)}(b\Omega)$. Let $\mathcal{F}_1$ be the M^{n-2}-dimensional linear span of all such u's. We extend $u\mapsto v_u(\emptyset)$ linearly to all $u\in\mathcal{F}_1$.

For $0\le q\le n-1$, we use induction on q to construct an M^{n-q-2}-dimensional subspace $\mathcal{F}_{q+1}$ of $\mathcal{F}_q$ with the properties that for any $u\in\mathcal{F}_{q+1}$, there exists $v_u(k_1,\dots,k_q)\in L^2_{(0,n-q-2)}(b\Omega)$ for all $\{k_1,\dots,k_q\}\subset\{1,\dots,n-1\}$ such that

(1) $v_u(k_1,\dots,k_q)$ depends linearly on u.
(2) $v_u(k_1,\dots,k_q)$ is skew-symmetric with respect to indices $K=(k_1,\dots,k_q)$.
(3) $\overline{\partial}_b v_u(K)=m\sum_{j=1}^q(-1)^j z^m_{k_j}v_u(K;\hat{k_j})+(-1)^{q+|K|}u(1,\dots,n;\hat{K})$ where $|K|=k_1+\dots+k_q$. The hat $\hat{}$ indicates deletion of indices beneath it from the indices preceding the semicolon in the same enclosing parenthesis.

We now show how to construct $\mathcal{F}_{q+1}$ and $v_u(k_1,\dots,k_q)$ for $u\in\mathcal{F}_{q+1}$ and $\{k_1,\dots,k_q\}\subset\{1,\dots,n-1\}$ once $\mathcal{F}_q$ has been constructed. For any $u\in\mathcal{F}_q$ and any $\{k_1,\dots,k_q\}\subset$

$\{1, \ldots, n-1\}$, write $K = (k_1, \ldots, k_q)$, and let

$$w_u(K) = m \sum_{j=1}^{q} (-1)^j z_{k_j}^m v_u(K; \hat{k_j}) + (-1)^{q+|K|} u(1, \ldots, n; \hat{K}).$$

Then

$$\begin{aligned}
\overline{\partial}_b w_u(K) &= m \sum_{j=1}^{q} (-1)^j z_{k_j}^m \overline{\partial}_b v_u(K; \hat{k_j}) + (-1)^{q+|K|} \overline{\partial}_b u(1, \ldots, n; \hat{K}) \\
&= m \sum_{j=1}^{q} (-1)^j z_{k_j}^m \big(m \sum_{1 \le i < j} (-1)^i z_{k_i}^m v_u(K; \hat{k}_j, \hat{k}_i) + m \sum_{j < i \le q} (-1)^{i-1} z_{k_i}^m v_u(K; \hat{k}_j, \hat{k}_i) \\
&\qquad - (-1)^{q+|K|-k_j} u(1, \ldots, n; \widehat{(K; \hat{k}_j)}) \big) + (-1)^{q+|K|} \overline{\partial}_b u(1, \ldots, n; \hat{K}) \\
&= (-1)^{q+|K|} \big(-m \sum_{j=1}^{q} (-1)^{j-k_j} z_{k_j}^m u(1, \ldots, n; \widehat{(K; \hat{k}_j)}) + \overline{\partial}_b u(1, \ldots, n; \hat{K}) \big) \\
&= (-1)^{q+|K|} \big(-m \sum_{j=1}^{q} z_{k_j}^m u(k_j, (1, \ldots, n; \hat{K})) + \overline{\partial}_b u(1, \ldots, n; \hat{K}) \big) = 0.
\end{aligned}$$

We again decompose $\mathcal{F}_q$ into a direct sum of M^{n-q-2} linear subspaces, each of which is M-dimensional. Since $\dim(S_{n-q-2}) < M$ and $\overline{\partial}_b w_u(K) = 0$, there exists a non-zero form u in each of these subspaces such that $\overline{\partial}_b v_u(K) = w_u(K)$ for some $v_u(K) \in L^2_{(0,n-q-2)}(b\Omega)$. Since $w_u(K)$ is skew-symmetric with respect to indices K, we may choose $v_u(K)$ to be skew-symmetric with respect to K as well. The subspace $\mathcal{F}_{q+1}$ of $\mathcal{F}_q$ is then the linear span of all such u's.

Note that $\dim(\mathcal{F}_{n-1}) = 1$. Let u be any non-zero form in $\mathcal{F}_{n-1}$ and let

$$g = w_u(1, \ldots, n-1) = m \sum_{j=1}^{n-1} z_j^m v_u(1, \ldots, \hat{j}, \ldots, n-1) - (-1)^{n + \frac{n(n-1)}{2}} u(n).$$

Then $g \in L^2(b\Omega)$ and $\overline{\partial}_b g = 0$. Therefore, g has a holomorphic extension G to Ω such that the non-tangential limit of G agrees with g a.e. on $b\Omega$ (cf. Theorem 7.1 in [Ky95]). By the *reductio ad absurdum* assumption, G extends holomorphically to $\widetilde{\Omega}$. Write $z' = (z_1, \ldots, z_{n-1})$. For sufficiently small $\varepsilon > 0$ and $\delta > 0$,

$$\begin{aligned}
&\int_{|x_n|<\varepsilon, |z'|<\varepsilon} \big| \big(G + (-1)^{n+\frac{n(n-1)}{2}} u(n)\big)(\delta z', x_n + ih(\delta z', x_n)) \big| dV(z') dx_n \\
&\qquad \le m\delta^m \sum_{j=1}^{n-1} \int_{|x_n|<\varepsilon, |z'|<\varepsilon} |z_j|^m |v_u(1, \ldots, \hat{j}, \ldots, n-1)(\delta z', x_n + ih(\delta z', x_n))| dV(z') dx_n \\
&\qquad \le m\delta^{m-2(n-1)} \varepsilon^m \sum_{j=1}^{n-1} \|v_u(1, \ldots, \hat{j}, \ldots, n-1)\|_{L^1(b\Omega)}.
\end{aligned}$$

Choosing $m > 2(n-1)$ and letting $\delta \to 0$, we obtain

$$G(0, x_n + ih(0, x_n)) = -(-1)^{n+\frac{n(n-1)}{2}} u(n)(0, x_n + ih(0, x_n)).$$

However, $u(n)(0, z_n)$ is a non-trivial linear combination of functions of form $1/z^k$ with k a positive integer. This leads to a contradiction with the analyticity of G near the origin. We therefore conclude the proof of Theorem 1.1.

4. Further Remarks

(1) The analogue of Theorem 1.1 for the $\overline{\partial}$-Neumann Laplacian $\Box_q$ also holds under the assumption that $\operatorname{int}(\operatorname{cl}(\Omega)) = \Omega$. This is a consequence of the sheaf cohomology theory (see [S53, L66, O88]), in light of Lemma 2.1. (We thank Professor Y.-T. Siu for drawing our attention to [L66], by which the construction here is inspired.) The above proof of Theorem 1.1 can be easily modified to give a proof of this $\overline{\partial}$-Neumann Laplacian analogue, bypassing sheaf cohomology arguments. In this case, one can actually choose m to be any positive integer, independent of the dimension n. The non-elliptic nature of $\overline{\partial}_b$-complex seems to require that the m in the above proof be dependent on n. It follows from Hörmander's L^2-existence theorem for the $\overline{\partial}$-operator that $\inf \operatorname{spec}(\Box_q) > 0$ for all $1 \le q \le n-1$ for any bounded pseudoconvex domain in $\mathbb{C}^n$ (see [H65, H91]). Therefore, for a bounded domain Ω in $\mathbb{C}^n$ such that $\operatorname{int}(\operatorname{cl}(\Omega)) = \Omega$, the following statements are equivalent: (a) Ω is pseudoconvex; (b) $\inf \operatorname{spec}(\Box_q) > 0$ for all $1 \le q \le n-1$; (c) $\inf \operatorname{esspec}(\Box_q) > 0$ for all $1 \le q \le n-1$.

(2) Let Ω be a bounded Lipschitz domain in $\mathbb{C}^n$ and let $p \ge 1$. Consider $\overline{\partial}_{b,q}\colon L^p_{(0,q)}(\Omega) \to L^p_{(0,q+1)}(b\Omega)$, $0 \le q \le n-2$, where $L^p_{(0,q)}(b\Omega)$ are boundary $(0,q)$-forms with L^p-coefficients. Let $\mathcal{K}_{n-1}$ be the space of all $f \in \operatorname{Dom}(\overline{\partial}_{b,n-1})$ such that

$$\int_{b\Omega} f \wedge \alpha = 0$$

for all $\alpha \in C^\infty_{(n,0)}(\overline{\Omega}) \cap \mathcal{N}(\overline{\partial})$. Let $H^p_q(b\Omega) = \mathcal{N}(\overline{\partial}_{b,q}) / \mathcal{R}(\overline{\partial}_{b,q-1})$, $1 \le q \le n-2$, and $H^p_{n-1}(b\Omega) = \mathcal{K}_{n-1} / \mathcal{R}(\overline{\partial}_{b,n-1})$. Then the proof of Theorem 1.1 implies that Ω is pseudoconvex if $\dim(H^p_q(b\Omega)) < \infty$ for all $1 \le q \le n-1$.

(3) The generalization to (p,q)-forms is trivial. We deal with $(0,q)$-forms only for economy of notations.

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